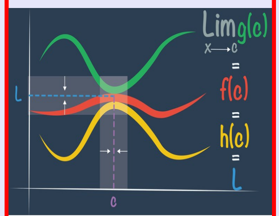
# Calculus I

# Lecture 14




Feb 19-8:47 AM

Class Quiz 12

$\frac{dA}{dt}$

How fast the area of a circle increases is its radius increase at $\frac{2}{\pi}$ cm/s when its radius is 5cm?

$\underbrace{\text{radius is 5cm}}_{r = 5\,\text{cm}}$

$\frac{dr}{dt} = \frac{2}{\pi}$ cm/s

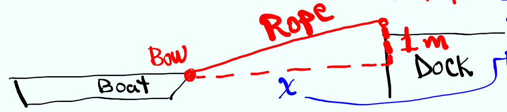
$A = \pi r^2$

$\frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\cancel{\pi} \cdot 5 \cdot \frac{2}{\cancel{\pi}} = \boxed{20\ \text{cm}^2/\text{s}}$

Jan 28-8:04 AM

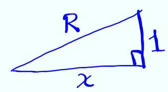
A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow on the boat.




$\frac{dR}{dt} = -1$ m/s

$\frac{dx}{dt} = ?$ $x=8$

If the rope is pulled in at the rate of 1 cm/s, how fast the boat approaching the dock when it is 8 m from the dock?




$x^2 + 1^2 = R^2$

$2x\frac{dx}{dt} + 0 = 2R\frac{dR}{dt}$

$x = 8$

$8^2 + 1^2 = R^2$

$R^2 = 65$

$R = \sqrt{65}$

$8\frac{dx}{dt} = \sqrt{65}\cdot(-1)$

$\frac{dx}{dt} = \frac{-\sqrt{65}}{8}$ m/s.

Jan 28-8:19 AM

The sum of two positive numbers is 16.

$x > 0,\ y > 0$

$x + y = 16$ $\quad y = 16 - x$

$y = 16 - 8$

$\boxed{y = 8}$

What is the smallest possible value of the sum of $\boxed{\text{their squares}}$?

$x^2 + y^2$

$8^2 + 8^2 =$

$\boxed{128}$

$f(x) = x^2 + (16-x)^2$

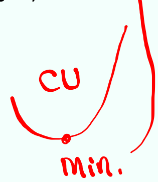
Minimize this

$f'(x) = 2x + 2(16-x)^1\cdot(-1)$

$= 2x - 32 + 2x$

$= 4x - 32$

$f''(x) = 4 > 0$ CU



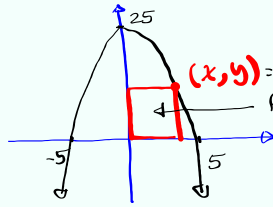

$f'(x) = 0$

$4x - 32 = 0$

$\boxed{x = 8}$

Jan 28-8:30 AM

Find the area of the largest rectangle in the first quadrant with one corner on the graph of $\boxed{y = 25 - x^2}$ and two sides are on x-axis & y-axis.




$(x,y) = (x, 25-x^2)$

$A = xy$

$A(x) = x(25-x^2)$

$A(x) = 25x - x^3$

Maximize

$A'(x) = 25 - 3x^2$

$A'(x) = 0$

$25 - 3x^2 = 0$

$3x^2 = 25$

$x^2 = \frac{25}{3}$

$x = \frac{5}{\sqrt{3}}$

$A''(x) = -6x < 0 \rightarrow CD$

QI $\rightarrow x > 0$

Max

$y = 25 - x^2$

$= 25 - \frac{25}{3}$

$\boxed{y = \frac{50}{3}}$

$A = xy$

$= \frac{5}{\sqrt{3}} \cdot \frac{50}{3} = \frac{250}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{250\sqrt{3}}{9}$ units$^2$

Jan 28-8:37 AM

Evaluate

$$\lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{1-4x} = \frac{\infty}{-\infty} \quad \text{I.F.}$$

Divide by $x$, as $x \to \infty$, $x = \sqrt{x^2}$

$$\lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{1-4x} = \lim_{x \to \infty} \frac{\frac{\sqrt{4x^2+1}}{x}}{\frac{1-4x}{x}} = \lim_{x \to \infty} \frac{\sqrt{\frac{4x^2+1}{x^2}}}{\frac{1-4x}{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{4 + \cancel{\frac{1}{x^2}}^{\to 0}}}{\cancel{\frac{1}{x}}^{\to 0} - 4} = \frac{\sqrt{4}}{-4} = \frac{2}{-4} = \boxed{-\frac{1}{2}}$$

Jan 28-8:47 AM

Evaluate $\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{1-4x} = \frac{\infty}{\infty}$ I.F.

Divide by $x$, as $x \to -\infty$, $x = -\sqrt{x^2}$

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{1-4x} = \lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2+1}}{x}}{\frac{1-4x}{x}} = \lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2+1}}{-\sqrt{x^2}}}{\frac{1-4x}{x}}$$

$$= -\lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2+1}{x^2}}}{\frac{1-4x}{x}} = -\lim_{x\to-\infty} \frac{\sqrt{4+\cancel{\frac{1}{x^2}}^{\to 0}}}{\cancel{\frac{1}{x}}^{\to 0}-4} = -\frac{\sqrt{4}}{-4}$$

$$= \boxed{\frac{1}{2}}$$

Jan 28-8:53 AM

Evaluate

$\lim_{x\to\infty} \left[\sqrt{x^2+2bx} - x\right] = \infty - \infty$ I.F.

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2bx} - x\right)\left(\sqrt{x^2+2bx} + x\right)}{\sqrt{x^2+2bx} + x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+2bx - \cancel{x^2}}{\sqrt{x^2+2bx} + x} = \lim_{x\to\infty} \frac{2bx}{\sqrt{x^2+2bx} + x} = \frac{\infty}{\infty}$$

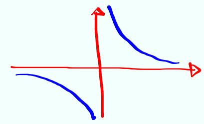
Divide by $x$, as $x \to \infty$, $x = \sqrt{x^2}$

$$= \lim_{x\to\infty} \frac{\frac{2bx}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{2bx}{x^2}} + \frac{x}{x}} = \lim_{x\to\infty} \frac{2b}{\sqrt{1+\cancel{\frac{2b}{x}}^{\to 0}} + 1}$$

$$= \frac{2b}{1+1} = \boxed{b}$$

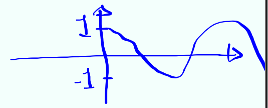
Jan 28-9:00 AM

Evaluate $\lim_{x \to 0} x^2 \cos \frac{1}{x}$



Recall from trig

$-1 \le \cos A \le 1$




$-1 \le \cos \frac{1}{x} \le 1$

Multiply by $x^2$ $\quad x^2 > 0$

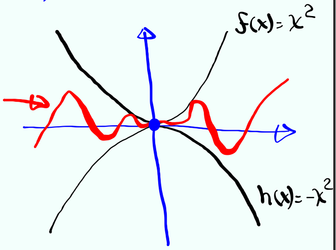
$-x^2 \le x^2 \cos \frac{1}{x} \le x^2$

$\lim_{x \to 0} (-x^2) = 0$

$\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$ by S.T.

$\lim_{x \to 0} x^2 = 0$

$g(x) = x^2 \cos \frac{1}{x} \longrightarrow$




Jan 28-9:11 AM

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Find $f(x)$ if $\quad f''(x) = 8x^3 + 5$

$f(1) = 0 \;, \quad f'(1) = 8.$

Hint: $8 = 2 \cdot 4$

$f''(x) = 2 \cdot \boxed{4x^3} + 5$

$f'(x) = 2x^4 + 5x + C$

$f'(1) = 2(1)^4 + 5(1) + C = 8 \rightarrow C = 1$

$f'(x) = 2x^4 + 5x + 1$

$f(x) = 2 \cdot \frac{x^5}{5} + 5 \cdot \frac{x^2}{2} + x + C$

$f(1) = \frac{2}{5} + \frac{5}{2} + 1 + C = 0$

$\frac{4 + 25 + 10}{10} + C = 0$

$\boxed{C = \frac{-39}{10}}$

$\boxed{f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}}$

Jan 28-9:20 AM

$f(0)=1$ , $f'(0)=2$ , $f''(0)=3$ ,

$f'''(x)=\cos x$ Find $f(x)$.

$f''(x)=\sin x+C$

$f''(0)=\cancel{\sin 0}^{\to 0}+C=3$ $\boxed{C=3}$

$f''(x)=\sin x+3$

$f'(x)=-\cos x+3x+C$

$f'(0)=-\cancel{\cos 0}^{\to 1}+\cancel{3(0)}^{\to 0}+C=2$ $\boxed{C=3}$

$f'(x)=-\cos x+3x+3$

$f(x)=-\sin x+3\cdot\frac{x^2}{2}+3x+C$

$f(0)=-\cancel{\sin 0}^{\to 0}+\cancel{\frac{3}{2}(0)}^{\to 0}+\cancel{3(0)}^{\to 0}+C=1$

$\boxed{C=1}$

$f(x)=-\sin x+\frac{3}{2}x^2+3x+1$

$f'(x)=-\cos x+\frac{3}{\cancel{2}}\cdot\cancel{2}x+3$

$f''(x)=-(-\sin x)+3=\sin x+3$

$f'''(x)=\cos x$

Jan 28-9:30 AM

Class QZ 13

$\boxed{\text{Find } f(x)}$ if $f'(x)=5x^4-3x^2+4$

and $f(-1)=4$ ✓

$f(x)=x^5-x^3+4x+C$

$f(-1)=(-1)^5-(-1)^3+4(-1)+C=4$

$-1+1-4+C=4$

$\boxed{C=8}$

Ans: $\boxed{f(x)=x^5-x^3+4x+8}$

Jan 28-9:41 AM

Rolle's Theorem

Consider a function $f(x)$ such that

1) $f(x)$ is continuous on $[a,b]$,

2) $f(x)$ is differentiable on $(a,b)$, and

3) $f(a) = f(b)$

then there is at least a number $c$ in $(a,b)$ such that $f'(c) = 0$

Jan 28-10:10 AM

Consider $f(x) = 3x^2 - 12x + 5$ and $[1,3]$

$f(x)$ is a polynomial function,

$f(x)$ is cont. on $[1,3]$

$f(x)$ is diff. on $(1,3)$

✓ $f(1) = 3(1)^2 - 12(1) + 5 = 3 - 12 + 5 = -4$ ✓

✓ $f(3) = 3(3)^2 - 12(3) + 5 = 27 - 36 + 5 = -4$ ✓

All conditions of the Rolle's thrm are met,

there is at least a number $c$ in $(1,3)$

such that $f'(c) = 0$

$f(x) = 3x^2 - 12x + 5$

$f'(x) = 6x - 12$

$f'(c) = 6c - 12 = 0$

$\boxed{c = 2}$

2 is in the interval $(1,3)$

Jan 28-10:13 AM

$f(x) = \sqrt{x} - \frac{1}{3}x$ , $[0,9]$

1) Is $f(x)$ cont. on $[0,9]$? ✓ Yes

Domain $[0,\infty)$

2) is $f(x)$ diff. on $(0,9)$? ✓ Yes

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

3) Is $f(0) = f(9)$? Yes

$f(0) = \sqrt{0} - \frac{1}{3}(0) = 0$

$f(9) = \sqrt{9} - \frac{1}{3}(9) = 3-3=0$

By Rolle's thrm, there is at least a number $c$ in $(0,9)$ such that $f'(c)=0$.

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$

$\frac{1}{2\sqrt{c}} = \frac{1}{3}$

$2\sqrt{c} = 3$

$\sqrt{c} = \frac{3}{2}$

$c = \left(\frac{3}{2}\right)^2 = \boxed{\frac{9}{4}}$

$\frac{9}{4}$ is in $(0,9)$

Jan 28-10:19 AM

Verify all conditions of the Rolle's thrm for $f(x) = x^3 - x^2 - 6x + 2$ on $[0,3]$ then find all numbers that satisfy the conclusion of the Rolle's thrm.

$f(x) = x^3 - x^2 - 6x + 2$

$f(x)$ is a polynomial function → Cont. & Diff. $(-\infty,\infty)$

$f(0) = 0^3 - 0^2 - 6(0) + 2 = 2$ ✓

$f(3) = 3^3 - 3^2 - 6(3) + 2 = 2$ ✓

By Rolle's thrm, there must be at least a number $c$ in $(0,3)$ such that $f'(c)=0$

$f'(x) = 3x^2 - 2x - 6$

$f'(c) = 3c^2 - 2c - 6 = 0$

$c = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} = \frac{2 \pm \sqrt{76}}{6}$

$c = \frac{2+\sqrt{76}}{6} \approx 1.786$ is in $(0,3)$ ✓

$c = \frac{2-\sqrt{76}}{6} \approx -1.120$

Jan 28-10:26 AM

Mean Value Theorem (MVT)

Consider a function $f(x)$ such that

1) It is Cont. on $[a,b]$, and

2) It is diff. on $(a,b)$

then there is at least a number $c$ in $(a,b)$

Such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Jan 28-10:36 AM

$f(x) = x^3 - 3x + 2$ , $[-2,2]$

$f(x)$ is a Polynomial Function $\rightarrow$ Cont. & Diff. $(-\infty,\infty)$

By MVT, there is at least a number $c$ in $(-2,2)$ Such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f(2) = 2^3 - 3(2) + 2 = 8 - 6 + 2 = 4$

$f(-2) = (-2)^3 - 3(-2) + 2 = -8 + 6 + 2 = 0$

$f'(x) = 3x^2 - 3$

$f'(c) = 3c^2 - 3$

$f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)}$

$3c^2 - 3 = \dfrac{4 - 0}{2 + 2}$

$3c^2 - 3 = \dfrac{4}{4}$

$3c^2 = 4$

$c^2 = \dfrac{4}{3}$ $c = \pm \dfrac{2}{\sqrt{3}}$

$\boxed{c \approx \pm 1.155}$

are in $(-2,2)$

Jan 28-10:40 AM

$f(x) = 2x^2 - 3x + 1$ , $[0,2]$

1) Check conditions of MVT.

Polynomial functions are cont. & diff. everywhere.

2) Find all numbers c that satisfy the conclusion of MVT.

$a=0$
$b=2$

$f(x) = 2x^2 - 3x + 1$

$f(0) = 1$ , $f(2) = 3$

$f'(x) = 4x - 3$

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$4c - 3 = \dfrac{3 - 1}{2 - 0}$

$4c - 3 = 1$

$4c = 4$

$\boxed{c = 1}$ is in $(0,2)$

Jan 28-10:46 AM

Check conditions of MVT for $f(x) = \frac{1}{x}$ on $[1,3]$, then find all numbers c that is in the conclusion of MVT.

$f(x) = \dfrac{1}{x}$

$f'(x) = \dfrac{-1}{x^2}$

Cont. ✓

Diff. ✓

$f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$

$\dfrac{-1}{c^2} = \dfrac{\frac{1}{3} - 1}{2}$

$\dfrac{-1}{c^2} = \dfrac{-\frac{2}{3}}{2}$

$\frac{2}{3}c^2 = 2$

$2c^2 = 6$

$c^2 = 3$ $c = \pm\sqrt{3}$

$\boxed{c = \sqrt{3}}$

$\sqrt{3}$ is in $(1,3)$

Jan 28-10:53 AM

Prove if $f(x)$ is diff. at $x=a$, then it is cont. at $x=a$.

We know: $f(x)$ is diff. at $x=a$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

we need to show: $\lim_{x\to a} f(x)=f(a)$

$$\lim_{x\to a} f(x)=\lim_{x\to a}\left[f(x)-f(a)+f(a)\right]$$

$$=\lim_{x\to a}\left[f(x)-f(a)\right]+\lim_{x\to a}f(a)$$

$$=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right]+\lim_{x\to a}f(a)$$

$$=\boxed{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}\cdot\boxed{\lim_{x\to a}(x-a)}+\boxed{\lim_{x\to a}f(a)}$$

$$=\underbrace{f'(a)\cdot(a-a)}_{0}+f(a)$$

$$=f(a)$$

$\lim_{x\to a} f(x)=f(a)$ therefore $f(x)$ is cont. at $x=a$.

Jan 28-11:03 AM

$\boxed{y=ax^3}$ , $\boxed{x^2+3y^2=b}$

Show these two curves are orthogonal.

Show product of derivatives is $-1$.

$$\frac{dy}{dx}=3ax^2$$

$$2x+6y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{6y}=\frac{-x}{3y}$$

$$\cancel{3}ax^2\cdot\frac{-x}{\cancel{3}y}=\frac{-ax^3}{y}=\frac{-y}{y}=\boxed{-1}$$

Curves are orthogonals.

$a=1$ , $b=1$

$y=x^3$ $\quad x^2+3y^2=1$

Jan 28-11:14 AM

Find $\boxed{\text{all points}}$ on the curve $x^2y^2+xy=2$ where slope of the tangent line is $-1$.

$\left.\frac{dy}{dx}\right|_{(\,,\,)}=-1$ $\quad \frac{-y}{x}=-1$ $\quad y=x$

$x^2y^2+xy=2$

$$2x\cdot y^2+\boxed{x^2\cdot 2y\cdot\frac{dy}{dx}}+1\cdot y+\boxed{x\cdot\frac{dy}{dx}}=0$$

$$(2x^2y+x)\frac{dy}{dx}=-2xy^2-y$$

$$\frac{dy}{dx}=\frac{-2xy^2-y}{2x^2y+x}=\frac{-y\cancel{(2xy+1)}}{x\cancel{(2xy+1)}}=\frac{-y}{x}$$

$x^2y^2+xy=2$

Since $y=x$

$x^2\cdot x^2+x\cdot x=2$

$x^4+x^2=2$

$x^4+x^2-2=0$

$(x^2+2)(x^2-1)=0$

$x^2+2=0$ has no real soln.

$x^2-1=0$

$x^2=1$

$x=\pm1$

$y=\pm1$

Points

$(1,1)$ & $(-1,-1)$

$y-1=-1(x-1)$ $\quad \boxed{y=-x+2}$

$y-(-1)=-1(x--1)$ $\quad \boxed{y=-x-2}$

Jan 28-11:23 AM